December 1995

Conformal Invariance, N-extended Supersymmetry and Massless Spinning Particles in Anti-de Sitter Space

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Abstract

Starting with a manifestly conformal ($O(d,2)$ invariant) mechanics model in d space and 2 time dimensions, we derive the action for a massless spinning particle in d-dimensional anti-de Sitter space. The action obtained possesses both gauge N-extended worldline supersymmetry and local $O(N)$ invarince. Thus we improve the old statement by Howe et al. that the spinning particle model with extended worldline supersymmetry admits only flat space-time background for $N > 2$ (spin greater one). The original $(d + 2)$-dimensional model is characterized by rather unusual property that the corresponding supersymmetry transformations do not commute with the conformal ones, in spite of the explicit $O(d,2)$ invariance of the action.

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1 Introduction

To describe a massless higher-spin particle moving in d-dimensional Minkowski space, Gershun and Tkach [1] and later Howe et al. [2] suggested the mechanics action with a gauge N-extended worldline supersymmetry and a local $O(N)$ invariance (for convenience, we refer to this theory as "GT model"); in Ref. [1] the massive case was completely treated too. For $d = 4$ it was thoroughly shown [2] that, upon quantization, the physical wave functions are subject to a relativistic conformally covariant equation for pure spin $\frac{1}{2}N$. This result can be simply generalized to $d \neq 4$ with a proper modification of notion "spin".

In the cases $N = 1, 2$, the GT model inevitably reduces to the previously developed mechanics systems for spin-$\frac{1}{2}$ [3] and spin-1 [4] particles respectively. The choices $N = 1$ and 2 were proved [2] to be the only ones, apart from the trivial spinless case, for which the GT model can be consistently extended to include coupling to an arbitrary gravitational background. As to $N > 2$, it was concluded in [2] that the only space-time background compartible with worldline supersymmetry is a flat one. In the present paper, we are going to improve the latter statement by explicitly showing that the GT model can be consistently generalized, for any N, to the case of space-times with constant non-zero curvature. The key observation for the existence of such a generalization of the model is its hidden conformal invariance.

As it was demonstrated in Ref. [2], for $D = 4$ the wave functions in the GT model satisfy the conformally covariant equation for a pure spin-$\frac{1}{2}N$ field strength (helicities $\pm\frac{1}{2}N$) [5]. That might apparently have implied conformal invariance of the model for all d and N. This proposal has been proved by Siegel [6], who found the ansatz to obtain the GT action from an explicitly conformal ($O(d,2)$ invariant) mechanics action in d space and 2 time dimensions; he has also shown [7] that all conformal wave equations, in all dimensions, can be derived in this way.

Siegel simply extended, to the higher-spin case, the construction used by Marnelius [8] to represent the actions for massless spin-0 and spin-$\frac{1}{2}$ particles in manifestly conformal form. It is worth briefly recalling here the spinless case. Let $x^a(\tau)$, $a = 0, 1, \ldots, d-1$, be some worldline in Minkowski space and $e(\tau)$ the associated einbein. The set of all histories $\{x^a(\tau), e(\tau)\}$ in Minkowski space turns out to be in a one-to-one correspondence with the family $\{Z^{\mathcal{A}}(\tau)\}$, $\mathcal{A} = d + 1, 0, 1, \ldots, d$, of trajectories passing through the domain

$$Z_+ \neq 0 \qquad Z_\pm = \frac{1}{\sqrt{2}}(Z^d \pm Z^{d+1}) \tag{1.1}$$

of the cone in $\mathbf{R}^{d,2}$

$$\eta_{\mathcal{AB}}Z^{\mathcal{A}}Z^{\mathcal{B}} = 0 \qquad \eta_{\mathcal{AB}} = \mathrm{diag}(- - + \ldots +) \ . \tag{1.2}$$

The correspondence reads

$$e = \left(\frac{1}{Z_+}\right)^2 \qquad x^a = \frac{Z^a}{Z_+} \tag{1.3}$$

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and its use reduces the $(d+2)$-dimensional Lagrangian

$$\mathcal{L} = \frac{1}{2}\dot{Z}^A\dot{Z}_A \qquad Z^A(\tau)Z_A(\tau) = 0 \qquad (1.4)$$

to that in Minkowski space

$$\mathcal{L} = \frac{1}{2e}\dot{x}^a\dot{x}_a \; . \qquad (1.5)$$

On the other hand, the reduction from $d+2$ to d dimensions could be carried out in two more inequivalent ways. First, we can restrict all trajectories to lie in the domain

$$Z^d \neq 0 \qquad (1.6)$$

of the cone (1.2). Then, the use of the ansatz (see, e.g., [9])

$$e = \left(\frac{r}{Z^d}\right)^2 \qquad y^A = r\frac{Z^A}{Z^d} \qquad A = d+1, 0, 1, \ldots, d-1 \qquad (1.7)$$

reduces the constrained model (1.4) to that of a massless particle in the anti-de Sitter (AdS) space

$$\eta_{AB}y^A y^B = -r^2 \; . \qquad (1.8)$$

Here $(-12r^{-2})$ is the curvature of the AdS space. Finally, restricting the dynamics to the domain

$$Z^{d+1} \neq 0 \qquad (1.9)$$

of the cone and setting

$$e = \left(\frac{r}{Z^{d+1}}\right)^2 \qquad y^{\underline{A}} = r\frac{Z^{\underline{A}}}{Z^d} \qquad \underline{A} = 0, 1, \ldots, d \qquad (1.10)$$

we deduce from (1.4) the massless mechanics model in the de Sitter (dS) space

$$\eta_{\underline{AB}}y^{\underline{A}} y^{\underline{B}} = r^2 \; . \qquad (1.11)$$

Below our strategy will be to apply the AdS ansatz (1.6)-(1.8), instead of the Minkowskian one (1.1), (1.2), for reducing the $(d+2)$-dimensional mechanics system of Ref. [6] to d space-time dimensions. This will lead to the mechanics action for a massless spinning particle, moving in the AdS space, which possesses both N-extended local supersymmetry and gauge $O(N)$ invariance. We restrict our attention to the AdS space leaving aside the dS one, because of the well-known fact that for $d = 4$ only the AdS symmetry algebra $so(3,2)$, and not the dS one $so(4,1)$, has unitary positive-energy representations [10, 11, 12] consistent with one-particle states interpretation. What is more, the AdS space can be supersymmetrized [13] and arises as a ground-state solution in extended supergravities (see, e.g., [14]).

The paper is organized as follows. In section 2 we introduce the locally supersymmetric mechanics system defined on a fibre bundle of the cone in $\mathbf{R}^{d,2}$ and describe

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three inequivalent reductions from $d+2$ to d dimensions. The spinning particle model in the AdS space is derived and the corresponding N-extended supersymmetry transformations are given in section 3. We present here two different formulations for the model: (i) as a globally $SO(d-1,2)$ invariant system described by constrained variables; (ii) as a mechanics system in the background curved space-time. The results of the paper and further perspectives are discussed in conclusion. In appendix we describe the technique to pass to internal coordinates on the AdS space.

2 $(d+2)$-dimensional model

We consider the mechanics system in $d+2$ dimensions with the action $S = \int d\tau \mathcal{L}$ given by

$$\mathcal{L} = \frac{1}{2}\dot{Z}^{\mathcal{A}}\dot{Z}_{\mathcal{A}} + \frac{i}{2}\Gamma_i{}^{\mathcal{A}}\dot{\Gamma}_{i\mathcal{A}} - \frac{i}{2}\varphi_{ij}\Gamma_i{}^{\mathcal{A}}\Gamma_{i\mathcal{A}} \,. \tag{2.1}$$

Here the bosonic $Z^{\mathcal{A}}(\tau)$ and fermionic $\Gamma_i{}^{\mathcal{A}}(\tau)$, $i = 1, \dots, N$, dynamical variables are subject to the constraints

$$Z^{\mathcal{A}}Z_{\mathcal{A}} = 0 \tag{2.2}$$
$$Z^{\mathcal{A}}\Gamma_{i\mathcal{A}} = 0 \tag{2.3}$$

and $\varphi_{ij}(\tau)$, $\varphi_{ij} = -\varphi_{ji}$, are Lagrange multipliers. Hence $Z^{\mathcal{A}}$ parametrize the cone in $\mathbf{R}^{d,2}$, whilst $\Gamma_i{}^{\mathcal{A}}$ form n tangent vectors to point Z of the cone.

Siegel [6] introduced the above constraints into the action, with the aid of appropriate Lagrange multipliers $\varphi(\tau)$ and $\varphi_i(\tau)$

$$\mathcal{L}' = \mathcal{L} - \frac{1}{2}\varphi Z^2 - i\varphi_i(Z, \Gamma_i). \tag{2.4}$$

We prefer, however, to work with the constrained variables from the very beginning, since Eqs. (2.2), (2.3) are crucial for making the reduction from $d+2$ to d dimensions.

In the Hamiltonian approach, the above system is specified by the first-class constraints [6]

$$Z^{\mathcal{A}}Z_{\mathcal{A}} = Z^{\mathcal{A}}P_{\mathcal{A}} = P^{\mathcal{A}}P_{\mathcal{A}} = 0$$
$$Z^{\mathcal{A}}\Gamma_{i\mathcal{A}} = P^{\mathcal{A}}\Gamma_{i\mathcal{A}} = 0$$
$$\Gamma_i{}^{\mathcal{A}}\Gamma_{j\mathcal{A}} = 0 \,. \tag{2.5}$$

Along with the explicit global $O(d,2)$ invariance (conformal invariance), the model possesses a rich gauge structure. The action remains unchanged under worldline reparametrizations

$$\delta Z^{\mathcal{A}} = \varepsilon \dot{Z}^{\mathcal{A}} - \frac{1}{2}\dot{\varepsilon}Z^{\mathcal{A}}$$
$$\delta \Gamma_i{}^{\mathcal{A}} = \varepsilon \dot{\Gamma}_i^{\mathcal{A}}$$
$$\delta \varphi_{ij} = \partial_\tau(\varepsilon \varphi_{ij}) \tag{2.6}$$

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and local $O(N)$ transformations

$$
\begin{aligned}
\delta Z^{\mathcal{A}} &= 0 \\
\delta \Gamma_i{}^{\mathcal{A}} &= \varepsilon_{ij} \Gamma_j{}^{\mathcal{A}} \\
\delta \varphi_{ij} &= \dot{\varepsilon}_{ij} + \varepsilon_{k[i}\varphi_{j]k} \, .
\end{aligned} \tag{2.7}
$$

Moreover, the action is invariant under local N-extended supersymmetry transformations of rather unusual form. These transformations involve an external $(d+2)$-vector $W^{\mathcal{A}}(\tau)$, chosen to satisfy the only requirement

$$
Z^{\mathcal{A}}(\tau) W_{\mathcal{A}}(\tau) \neq 0 \tag{2.8}
$$

for the worldline $\{Z^{\mathcal{A}}(\tau), \Gamma_i{}^{\mathcal{A}}(\tau), \varphi_{ij}(\tau)\}$ in field, and read as follows

$$
\begin{aligned}
\delta Z^{\mathcal{A}} &= i\alpha_i \Gamma_i{}^{\mathcal{A}} \\
\delta \Gamma_i{}^{\mathcal{A}} &= Z^{\mathcal{A}} \overset{\bullet}{\alpha}_i - \dot{Z}^{\mathcal{A}} \alpha_i + \frac{i}{(Z,W)} \Gamma_i{}^{\mathcal{B}} \Gamma_{j\mathcal{B}} \alpha_j W^{\mathcal{A}} \\
\delta \varphi_{ij} &= -\frac{i}{(Z,W)} \alpha_{[i} \overset{\bullet}{\Gamma}_{j]}{}^{\mathcal{A}} W_{\mathcal{A}} \, .
\end{aligned} \tag{2.9}
$$

Here $\overset{\bullet}{\alpha}_i$ denotes the $O(N)$-covariant derivative of α_i

$$
\overset{\bullet}{\alpha}_i = \dot{\alpha}_i - \varphi_{ij}\alpha_j \tag{2.10}
$$

and similarly for $\overset{\bullet}{\Gamma}_i{}^{\mathcal{A}}$. The origion of the last term in $\delta\Gamma$ is to preserve the constraint (2.3)

$$
(\delta Z, \Gamma_i) + (Z, \delta\Gamma_i) = 0 \, . \tag{2.11}
$$

Varying (2.9) with respect to $W^{\mathcal{A}}$ leads to the transformations

$$
\delta Z^{\mathcal{A}} = 0 \qquad \delta\Gamma_i{}^{\mathcal{A}} = i(\Gamma_i, \Gamma_j)\sigma_j{}^{\mathcal{A}} \qquad \delta\varphi_{ij} = -i(\sigma_{[i}, \overset{\bullet}{\Gamma}_{j]}) \tag{2.12}
$$

with the parameter $\sigma_i{}^{\mathcal{A}}(\tau)$ constrained by

$$
(Z, \sigma_i) = 0 \, . \tag{2.13}
$$

This is a trivial gauge invariance of the form

$$
\delta\phi^i = \Omega^{ij} \frac{\delta S[\phi]}{\delta\phi^i} \qquad \Omega^{ij} = -\Omega^{ji}
$$

each action $S[\phi^i]$ of (bosonic) variables ϕ^i possesses.

The expressions (2.9) become W-independent only on the mass shell. Off-shell, however, the supersymmetry transformations do not commute with the conformal ones, in spite of the manifest $O(d,2)$ invariance of $\mathcal{L}$! It should be pointed out that

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the supersymmetry transformations in the model (2.4) do not involve W and, hence, commute with the conformal ones (see Eq. (3.3b) in [6]). But such a dependence on W turns out to be inevitable upon putting forward the equations of motion (2.2) and (2.3) for φ and φ_i, respectively. This dependence implies in fact that there is no universal form for the local supersymmetry transformations in those regions of the extended cone (2.2), (2.3) that lead to the different d-dimensional space-times: Minkowski, anti-de Sitter and de Sitter ones.

Without loss of generality, $W^{\mathcal{A}}$ may be taken to be τ-independent. There are three inequivalent choices: light-like

$$W^{\mathcal{A}}_{(Mink)} = (-\frac{1}{\sqrt{2}}, 0, \ldots, 0, \frac{1}{\sqrt{2}}) \tag{2.14}$$

space-like

$$W^{\mathcal{A}}_{(AdS)} = (0, \ldots, 0, \frac{1}{r}) \tag{2.15}$$

and time-like

$$W^{\mathcal{A}}_{(dS)} = (\frac{1}{r}, 0, \ldots, 0) . \tag{2.16}$$

Now, the spinless reduction schemes, we have reviewed in sec.1, are described uniformly as follows. With a given $W^{\mathcal{A}}$ we associate the unique domain of the cone that is specified by the condition

$$\frac{1}{e} = (Z, W)^2 > 0 \tag{2.17}$$

and can be parametrized by e and the projective variables

$$\frac{Z^{\mathcal{A}}}{(Z, W)} \tag{2.18}$$

of which only d ones are independent. The stability group of $W^{\mathcal{A}}$ proves to be the symmetry group of the corresponding space-time. In the case of non-zero spin $(N \neq 0)$ we naturally decompose $\Gamma_i{}^{\mathcal{A}}$ by the rule

$$\lambda_i = e(\Gamma_i, W) \tag{2.19}$$

$$\Psi_i{}^{\mathcal{A}} = \Gamma_i{}^{\mathcal{A}} - (\Gamma_i, W)\frac{Z^{\mathcal{A}}}{(Z, W)} . \tag{2.20}$$

In accordance with Eqs. (2.2) and (2.3), we have

$$(Z, \Psi_i) = (W, \Psi_i) = 0 \tag{2.21}$$

as well as

$$\Gamma_i{}^{\mathcal{A}}\Gamma_{j\mathcal{A}} = \Psi_i{}^{\mathcal{A}}\Psi_{j\mathcal{A}} . \tag{2.22}$$

Let us illustrate the reduction procedure on the example of Minkowski space, considered previously in Ref. [6], for which $W^{\mathcal{A}}$ is given by Eq. (2.4) or, equivalently,

$$W_- = 1 \qquad W_+ = W^a = 0 . \tag{2.23}$$

Now, the variables x^a (1.3), $a = 0, 1, \ldots, d-1$, form the native unconstrained subset of (2.18). For $\Psi_i{}^{\mathcal{A}}$ we get

$$
\begin{aligned}
\Psi_{i+} &= 0 \qquad \Psi_{i-} = -x^a \Psi_{ia} \\
\Psi_i{}^a &= \Gamma_i{}^a - \frac{1}{e}\lambda_i x^a
\end{aligned}
\tag{2.24}
$$

$\Psi_i{}^a$ being unconstrained. The $\mathcal{L}$ (2.1) turns into

$$
\mathcal{L}_{Mink} = \frac{1}{2e}\dot{x}^a \dot{x}_a + \frac{i}{2}\Psi_i{}^a \dot{\Psi}_{ia} - \frac{i}{e}\lambda_i \Psi_i{}^a \dot{x}_a - \varphi_{ij}\Psi_i{}^a \Psi_{ja}
\tag{2.25}
$$

and after the redefinition

$$
\varphi_{ij} = f_{ij} + \frac{i}{e}\lambda_i \lambda_j
\tag{2.26}
$$

it takes the standard form [1, 2]

$$
\mathcal{L}_{Mink} = \frac{1}{2e}(\dot{x}^a - i\lambda_i \Psi_i{}^a)(\dot{x}_a - i\lambda_j \Psi_{ja}) + \frac{i}{2}\Psi_i{}^a(\dot{\Psi}_{ia} - f_{ij}\Psi_{ja}) .
\tag{2.27}
$$

Finally, the transformation rules (2.9) can be shown to be equivalent to

$$
\begin{aligned}
\delta x^a &= i\alpha_i \Psi_i{}^a \\
\delta \Psi_i{}^a &= -\frac{1}{e}\alpha_i(\dot{x}^a - i\lambda_j \Psi_j{}^a) \\
\delta e &= 2i\lambda_i \alpha_i \qquad \delta \lambda_i = \dot{\alpha}_i - f_{ij}\alpha_j \qquad \delta f_{ij} = 0 .
\end{aligned}
\tag{2.28}
$$

These are exactly the supersymmetry transformations in the GT model [1, 2].

3 Massless spinning particle model in the AdS space

We proceed to deriving the model for a massless spinning particle in the AdS space. Here the reduction is dictated by the choice (2.15). In accordance with Eqs. (2.19)-(2.21), we get

$$
\begin{aligned}
\lambda_i &= \frac{1}{r}e\Gamma_i{}^d \\
\Psi_i{}^A &= \Gamma_i{}^A - \frac{1}{r}y^A \Gamma_i{}^d \qquad \Psi_i{}^d = 0
\end{aligned}
\tag{3.1}
$$

where y^a and e are defined as in Eq. (1.7). The bosonic y^A and fermionic $\Psi_i{}^A$ degrees of freedom are constrained by

$$
\begin{aligned}
y^A y_A &= -r^2 \\
y^A \Psi_{iA} &= 0 .
\end{aligned}
\tag{3.2}
\tag{3.3}
$$

Thus Ψ_i present themselves N tangent vectors to point y of the AdS hyperboloid. Now, redefining φ_{ij} in (2.1) by the rule (2.26), the Lagrangian turns into

$$\mathcal{L}_{AdS} = \frac{1}{2e}(\dot{y}^A - i\lambda_i\Psi_i{}^A)(\dot{y}_A - i\lambda_j\Psi_{ja}) + \frac{i}{2}\Psi_i{}^A(\dot{\Psi}_{iA} - f_{ij}\Psi_{jA}) \ . \tag{3.4}$$

In the Hamiltonian approach, our system is completely characterized by the second-class

$$y^A y_A + r^2 = y^A p_A = 0 \qquad y^A \Psi_{iA} = 0 \tag{3.5}$$

and first-class

$$p^A p_A = 0 \qquad p^A \Psi_{iA} = 0 \qquad \Psi_i{}^A \Psi_{iA} = 0 \tag{3.6}$$

constraints.

The model (3.4) is manifestly invariant under the AdS symmetry group $O(d-1,2)$. It also possesses the invariance with respect to arbitrary worldline reparametrizations

$$
\begin{aligned}
\delta y^A &= \varepsilon\dot{y}^A \qquad \delta\Psi_i{}^A = \varepsilon\dot{\Psi}_i^A \\
\delta e &= \partial_\tau(\varepsilon e) \qquad \delta\lambda_i = \partial_\tau(\varepsilon\lambda_i) \qquad \delta f_{ij} = \partial_\tau(\varepsilon f_{ij})
\end{aligned}
\tag{3.7}
$$

and local $O(N)$ transformations

$$
\begin{aligned}
\delta y^A &= \delta e = 0 \\
\delta\Psi_i{}^A &= \varepsilon_{ij}\Psi_i{}^A \qquad \delta\lambda_i = \varepsilon_{ij}\lambda_j \\
\delta f_{ij} &= \dot{\varepsilon}_{ij} + \varepsilon_{k[i}f_{j]k}
\end{aligned}
\tag{3.8}
$$

that can be read off from Eqs. (2.6) and (2.7) respectively. From (2.9) we deduce the following supersymmetry transformations

$$
\begin{aligned}
\delta y^A &= i\alpha_i\Psi_i{}^A \\
\delta\Psi_i{}^A &= -\frac{1}{e}\alpha_i(\dot{y}^A - i\lambda_j\Psi_j{}^A) - \frac{i}{r^2}y^A\Psi_i{}^B\Psi_{jB}\alpha_j \\
\delta e &= 2i\lambda_i\alpha_i \qquad \delta\lambda_i = \dot{\alpha}_i - f_{ij}\alpha_j \\
\delta f_{ij} &= -\frac{i}{r^2}\alpha_{[i}\Psi_{j]A}\dot{y}^A \ .
\end{aligned}
\tag{3.9}
$$

It is of interest to reformulate the model in terms of internal (unconstrained) coordinates x^m, $m = 0, 1, \ldots, d-1$, on the AdS space. This may be most simply done with the aid of the technique developed in Ref. [15] and described briefly in Appendix. Then $\mathcal{L}_{AdS}$ takes the form

$$
\begin{aligned}
L &= \frac{1}{2e}g_{mn}(\dot{x}^m - i\lambda_i\psi_i{}^a e_a{}^m)(\dot{x}^n - i\lambda_j\psi_j{}^b e_b{}^n) \\
&+ \frac{i}{2}\psi_i{}^a(\dot{\psi}_{ia} - f_{ij}\psi_{ja} + \dot{x}^m\omega_{ma}{}^b\psi_{ib}) \ .
\end{aligned}
\tag{3.10}
$$

Here g_{mn} is the metric of the AdS space, the unconstrained fermionic variables $\psi_i{}^a$, transforming in the vector representation of the local Lorentz group, are defined by Eq. (A.2). The supersymmetry transformations (2.9) turn into

$$
\begin{aligned}
\delta x^m &= \mathrm{i}\alpha_i \psi_i{}^a e_a{}^m \\
\delta \psi_i{}^a &= -\frac{1}{e}\alpha_i(\dot{x}^m e_m{}^a - \mathrm{i}\lambda_j \psi_j{}^a) + \mathrm{i}\alpha_j \psi_j{}^b \psi_i{}^c e_b{}^m \omega_{mc}{}^a \\
\delta e &= 2\mathrm{i}\lambda_i \alpha_i \qquad \delta \lambda_i = \dot{\alpha}_i - f_{ij}\alpha_j \\
\delta f_{ij} &= -\frac{\mathrm{i}}{r^2}\alpha_{[i}\psi_{j]a}\dot{x}^m e_m{}^a
\end{aligned}
\qquad (3.11)
$$

where $e_m{}^a$ and ω_{mab} are the vierbein and torsion-free spin connection, respectively, see Appendix.

The theory with the Lagrangian (3.10) may be treated as a curved-space extension of that (2.27) in Minkowski space. However, the curved-space action proves to possess local N-extended worldline supersymmetry if and only if the space-time curvature is constant. In the case of negative constant curvature, i.e. the AdS geometry, we have

$$
\mathcal{R}_{abcd} = -\frac{1}{r^2}(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}) \qquad (3.12)
$$

and the corresponding supersymmetry transformations are given by Eq. (3.11).

Now, we are in a position to comment on incorrectness of the conclusion by Howe et al. [2] that their curved-space mechanics action admits only flat space-time background for $N > 2$. Howe et al. considered the mechanics system in a curved space with the Lagrangian

$$
L' = L - \frac{1}{4}\psi_i{}^a \psi_i{}^b \psi_j{}^c \psi_j{}^d \mathcal{R}_{abcd} \qquad (3.13)
$$

$\mathcal{R}_{abcd}$ being the curvature tensor of the space-time. They showed that for $N = 1, 2$ the action is invariant under supersymmetry transformations similar to (3.11) but with $\delta f_{ij} = 0$. They also pointed out that for $N > 2$ such transformations do not leave the action invariant unless $\mathcal{R}_{abcd} = 0$. However, since the curvature of the AdS space has the form (3.18), the structure in L' proportional to $\mathcal{R}$ can be absorbed by redefining f_{ij}

$$
-\frac{\mathrm{i}}{2}f_{ij}\psi_i{}^a \psi_{ja} - \frac{1}{4}\psi_i{}^a \psi_i{}^b \psi_j{}^c \psi_j{}^d \mathcal{R}_{abcd} = -\frac{\mathrm{i}}{2}f'_{ij}\psi_i{}^a \psi_{ja} \qquad (3.14)
$$

where

$$
f'_{ij} = f_{ij} - \frac{\mathrm{i}}{r^2}\psi_i{}^a \psi_{ja} . \qquad (3.15)
$$

After that one obtains the action $S = \int \mathrm{d}\tau L$ which have been shown to be invariant under (3.11).

4 Conclusion

In this paper we have suggested the action for a massless spinning particle in d-dimensional AdS space. The action possesses the invariance under local N-extended worldline superymmetric and $O(N)$ transformations. The mechanics system obtained presents itself the generalization of the GT model to the AdS space. Since our system is obtained via the reduction of the conformal model (2.1), similar to the GT model, its quantization can be fulfiled in the same way as it was done in Ref. [6].

We have pointed out very interesting property of the $(d+2)$-dimensional model (2.1) underlying our construction. Namely, the action functional is manifestly $O(d,2)$ invariant. But, due to the constraints on the dynamical variables, the local supersymmetry transformations do not commute with the conformal ones off-shell unless $N=1$. This remarkable property was not noted in Ref. [6].

Our discussion was restricted to the massless case. Most likely that the case of a massive spinning particle in d-dimensional AdS space can treated analogously to that it was done in Ref. [1] for Minkowski space, i.e. by introducing mass via reduction of the massless model in $d+1$ dimensions (see also [16]).

Similarly to the flat case, the supersymmetry transformations in the AdS space form an open algebra for $N>1$. Recently, Gates and Rana [17] have found an off-shell formulation for theories of spinning particle propagating in Minkowski space. It would be of interest to extend their results for the model developed in our paper.

A Reduction to internal coordinates on the AdS space

In this appendix we describe the reduction technique to internal coordinates on the AdS space denoted below M^d.

We first note the existence of a smooth mapping

$$G : M^d \to SO(d-1,2) \tag{A.1}$$

such that $G(y)$ moves a point (y, Ψ_i) of the surface (3.2), (3.3) to $(\mathbf{y}, \psi_i)$ of the form

$$
\begin{aligned}
\mathbf{y}^A &= G^A{}_B(y) y^B = (r, 0, \ldots, 0) \\
\psi_i{}^A &= G^A{}_B(y) \Psi_i{}^B = (0, \psi_i{}^a) \qquad a = 0, 1, \ldots, d-1 \ .
\end{aligned}
\tag{A.2}
$$

Here $\psi_i{}^a$ are unconstrained Grassmann variables. The choice of $G(y)$ is not unique. Such a mapping can be equally well replaced by another one

$$G'^A{}_B(y) = \Lambda^A{}_C(y) G^C{}_B(y) \tag{A.3}$$

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where Λ takes it values in the stability group of the marked point $\mathbf{y}$

$$\Lambda^A{}_B(y)\mathbf{y}^B = \mathbf{y}^A \qquad (\text{A.4})$$

and has the general structure

$$\Lambda : M^d \to SO(d-1,2)$$

$$\Lambda^A{}_B(y) = \begin{pmatrix} 1 & \vdots & 0 \\ . & \vdots & \ldots\ldots \\ 0 & \vdots & \Lambda^a{}_b(y) \end{pmatrix} \qquad \Lambda^a{}_b(y) \in SO(d-1,1)\ .$$

The set of all such mappings forms an infinite-dimensional group isomorphic to a local Lorentz group of the AdS space. This group acts on the surface (3.2), (3.3) by the law

$$(y, \Psi_i) \longrightarrow (y, G^{-1}(y)\Lambda(y)G(y)\Psi_i) \qquad (\text{A.5})$$

As is obvious, ψ_i transform as d-vectors with respect to the local Lorentz group.

Let x^m, $m = 0, 1, \ldots, d-1$, be local coordinates on the surface (3.2). The induced metric $\mathrm{d}s^2 = \eta_{AB}\mathrm{d}y^A\mathrm{d}y^B$ reads

$$\mathrm{d}s^2 = g_{mn}(x)\mathrm{d}x^m\mathrm{d}x^n \qquad (\text{A.6})$$

g_{mn} being a metric of constrant negative curvature $\mathcal{R} = -12/r^2$. Associated with $G(y)$ is a vierbein $e_m{}^a(x)$ of the metric that converts curved-space indices into flat-space ones. Really, let us define

$$e_m{}^A \equiv G^A{}_B\frac{\partial y^B}{\partial x^m} = -\frac{\partial G^A{}_B}{\partial x^m}y^B = (0, e_m{}^a) \qquad (\text{A.7})$$

where we have used the identity

$$G^{d+1}{}_B = -\frac{1}{r}y_B. \qquad (\text{A.8})$$

Since $G(y)$ belongs to $SO(d-1,2)$, one readily gets the relations

$$g_{mn} = e_m{}^a e_n{}^b \eta_{ab}. \qquad (\text{A.9})$$

By construction, the functions $x(y)$ and $\mathcal{G}(y)$ are defined only on the AdS hyperboloid. They can be uniquely extended onto the subspace of $\mathbf{R}^{d-1,2}$

$$U = \{y \in \mathbf{R}^{d-1,2}, \quad y^2 < 0\} \qquad (\text{A.10})$$

if one restricts them to have zeroth order of homogeneity in y

$$\frac{\partial x^m}{\partial y^C}y^C = 0 \qquad \frac{\partial \mathcal{G}^A{}_B}{\partial y^C}y^C = 0. \qquad (\text{A.11})$$

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Thus the variables x^m and σ, $\sigma \equiv (-y^A y_A)^{-1/2}$, can be chosen to parametrize U instead of y^A. Introducing

$$e_A{}^m \equiv \frac{\partial x^m}{\partial y^B}(G^{-1})^B{}_A = (0, e_a{}^m) \tag{A.12}$$

one finds

$$e_a{}^m e_m{}^b = \delta_a^b \tag{A.13}$$

therefore $e_a{}^m(x)$ is the inverse vierbein. Another geometric object, the torsion-free spin connection $\omega_{mab}(x)$, defined by

$$\omega_{mab} = -\omega_{mba} \qquad T_{mn}{}^a = \partial_n e_m{}^a - \partial_m e_n{}^a + \omega_n{}^a{}_b e_m{}^b - \omega_m{}^a{}_b e_n{}^b = 0$$

can be represented in the form

$$\omega_m{}^a{}_b = G^a{}_C \frac{\partial (G^{-1})^C{}_b}{\partial x^m}. \tag{A.14}$$

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